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ELASTOPLASTIC TORSION OF A CYLINDRICAL ROD FOR FINITE DEFORMATIONS*

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A solution of the problem of the torsion of a cylindrical rod was obtained in /1/ for a general, isotropic, incompressible elastic material. The present paper gives an analytical solution of the elastoplastic torsion problem for finite deformations, written in terms of quadratures of elliptic functions. The non-linear kinematics of elastoplastic deformation is introduced into the defining equations with the help of a multiplicative decomposition of the deformation gradient into elastic and plastic components /2, 3/. The elastic deformation and rate of plastic deformation are related to the state of stress of the body, in accordance with the defining Mooney-Rivlin equations /4/ and the law of flow for finite deformations associated with the Tresca yield condition /5/. A non-linear first-order partial differential equation and the initial data at the elastoplastic boundary are obtained in order to determine the angle of rotation within the plastic zone of the basis formed from the eigenvectors of the stress tensor, relative to the radial direction. The integration of the resulting equation is reduced to determining the general integral of the Ricatti equation with right-hand side determined from the angular velocity of flow of the material within the plastic zone. It is shown that neglecting the finiteness of the deformation leads to too high an estimate of the rigidity of the rod.

1. Non-linear kinematics of elastoplastic deformation. The deformation of an elastoplastic body from its natural (undeformed) state is described by the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, \lambda)$ where $\mathbf{x}$ denotes the position occupied by the point $\mathbf{X}$ of the reference configuration after the deformation, and λ is the loading parameter. We introduce the following notation: $\mathbf{F} = \operatorname{Grad} \mathbf{x}$ is the deformation gradient, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green deformation tensor, $\mathbf{V} = \mathbf{B}^{1/2}$ is the left tensile tensor, $\mathbf{v}(\mathbf{x}, \lambda)$ is the spatial velocity field and $\mathbf{L} = \operatorname{grad} \mathbf{v}$ is the spatial velocity gradient. A dot denotes material differentiation with respect to λ.

We also introduce, as the basic kinematic characteristics of the elastoplastic deformation, the elastic deformation gradient $\mathbf{F}^e$ and plastic deformation gradient $\mathbf{F}^p$ /2, 3/. The separation of the total deformation into its elastic and plastic component is based on the idea of the configuration of the elastoplastic body completely free of internal stresses (such a configuration is placed mentally in 1:1 correspondence with every actual configuration and is determined by the mapping $\mathbf{x} = \mathbf{x}(\mathbf{p}, \lambda)$). Since the stresses in this configuration are equal to zero, it follows that the mapping $\mathbf{p} = \mathbf{p}(\mathbf{X}, \lambda)$ determines a purely plastic deformation, while the mapping $\mathbf{x} = \mathbf{x}(\mathbf{p}, \lambda)$ determines a purely elastic one. The rule of differentiation of a composite mapping leads to multiplicative decomposition of the deformation gradient /2/

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{1.1}$$

The factors $\mathbf{F}^e$ and $\mathbf{F}^p$ are not defined single-valuedly since an arbitrary local rotation of the elements in the configuration completely free from internal stresses again yields a configuration in which every element of the body is load-free. Therefore we can assume without loss of generality that the polar expansion of the Cauchy tensor $\mathbf{F}^e$ does not contain an orthogonal multiplier /3/, i.e. $\mathbf{F}^e = \mathbf{V}^e$. Differentiating with respect to λ the multiplicative decomposition of the deformation gradient and taking into account the fact that $\mathbf{L} = \mathbf{F}^{\cdot}\mathbf{F}^{-1}$ /4/, we obtain

$$\mathbf{L} = \mathbf{F}^{e\cdot}\mathbf{F}^{e-1} + \mathbf{F}^e\mathbf{F}^{p\cdot}\mathbf{F}^{p-1}\mathbf{F}^{e-1} \tag{1.2}$$

We introduce the tensor $\mathbf{L}^p$ which is a spatial plastic deformation rate gradient: $\mathbf{L}^p = \mathbf{F}^{p\cdot}\mathbf{F}^{p-1}$. The symmetric part of the tensor $\mathbf{L}^p$ is denoted by $\mathbf{D}^p$ and is called the plastic deformation rate tensor. Solving (1.2) for $\mathbf{L}^p$ and taking the symmetric parts from both sides of the resulting tensor equation, we obtain

$$\mathbf{D}^p = \operatorname{sym}[\mathbf{V}^{e-1}(\mathbf{L} - \mathbf{V}^{e\cdot}\mathbf{V}^{e-1})\mathbf{V}^e] \tag{1.3}$$

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where $\operatorname{sym} \mathbf{A}$ is the symmetric part of the tensor $\mathbf{A}$. Using (1.3) we introduce the kinematics of elastoplastic deformation into the defining equation of elastoplastic media.

The kinematics of elastoplastic deformation were first constructed in /5/. The construction was based on the additive decomposition of the deformation tensor which is defined in the accompanying (convective) coordinate system as the difference of the metric coefficients corresponding to the reference and the actual configuration. The same approach was used in /6/.

2. ***Complete relations between the velocity field and the stresses in an elastoplastic medium.*** The yield condition for an ideal elastoplastic material has the form

$$f(\sigma) = k \tag{2.1}$$

where σ is the Cauchy stress tensor and k is the yield point. We have the following defining equation in the Rivlin form /4/ for the points of the body which deforms elastically for all values of the load parameter λ up to the yield point, under the conditions of incompressibility:

$$\sigma = -p\mathbf{I} + \Sigma_1\mathbf{B} + \Sigma_{-1}\mathbf{B}^{-1}, \quad f(\sigma) < k \tag{2.2}$$

where $\mathbf{I}$ is the unity tensor, p is the hydrostatic pressure and Σ_1, Σ_{-1} are the coefficients of reaction of a hyperelastic material. In the region of plastic flow $(f(\sigma) = k)$ the total deformation gradient is written in the form (1.1). The elastic tensor $\mathbf{F}^e$ is determined in terms of the stress tensor from an equation analogous to (2.2)

$$\sigma = -p\mathbf{I} + \Sigma_1\mathbf{F}^{e2} + \Sigma_{-1}(\mathbf{F}^{e2})^{-1} \tag{2.3}$$

The law of flow under finite deformations associated with yield condition (2.1) has the form /5/ (Λ is an undetermined multiplier)

$$\mathbf{D}^p = \Lambda\partial f/\partial\sigma \tag{2.4}$$

From (1.3) and (2.4) it follows

$$\operatorname{sym}[\mathbf{F}^{e-1}(\mathbf{L} - \mathbf{F}^{e\cdot}\mathbf{F}^{e-1})\mathbf{F}^e] = \Lambda\partial f/\partial\sigma \tag{2.5}$$

We can obtain the complete relations between the velocity field and the stresses in an elastoplastic medium by inverting (2.3) and substituting the result into (2.5) /7/. We note that the Euler velocity field occurs in the complete relations through the gradient $\mathbf{L}$, and in the form of a convective term in the expression for the material derivative $\mathbf{F}^{e\cdot}$. In what follows we shall assume that the coefficients of reaction of a hyperelastic material are constants (a Mooney-Rivlin material /4/): $\Sigma_1 = \mu(^1/_2 + \beta)$, $\Sigma_{-1} = -\mu(^1/_2 - \beta)$, $\mu > 0$, $|\beta| \leqslant {}^1/_2$. We will use, as the yield condition, the Tresca criterion of the greatest tangential stress.

We also note that Eqs.(2.4) and (2.5) hold only if active loading takes place at the yield point

$$f(\sigma) = k, \quad \operatorname{tr}[(\partial f/\partial\sigma)\sigma^{\cdot}] = 0$$

In the case when $f(\sigma) < k$, or

$$f(\sigma) = k, \quad \operatorname{tr}[(\partial f/\partial\sigma)\sigma^{\cdot}] < 0$$

(i.e. in the last case we have the unloading after the elastoplastic state has been reached), the multiplier Λ is equal to zero. Thus the complete relations between the velocity field and the stresses during unloading have the form

$$\operatorname{sym}[\mathbf{F}^{e-1}(\mathbf{L} - \mathbf{F}^{e\cdot}\mathbf{F}^{e-1})\mathbf{F}^e] = 0$$

where we should replace the tensor $\mathbf{F}^e$ by its expression in terms of the Cauchy stress tensor σ in accordance with (2.3).

3. ***Finite elastoplastic deformations of a cylindrical rod under torsion.*** Let us consider a cylindrical rod made of incompressible elastoplastic material. We will denote by R_0 the radius of cross-section of the rod in its natural (unstressed) state, and we shall use this state as reference.

The purely elastic deformation of a rod under torsion is determined by the mapping /1, 4/: $r = E^{-1/2}R$, $\theta = \Theta + DZ$, $r = EZ$. Here R, Θ, Z are the cylindrical coordinates in the reference configuration and in space respetively, D is the torsion and E is the elongation. The physical non-zero components of the Cauchy stress tensor are given by the formulas

$$\begin{gathered}\sigma_{rr} = {}^1/_2\Sigma_1D^2r^2 + A, \quad \sigma_{\theta\theta} = \sigma_{rr} + \Sigma_1D^2r^2 \\ \sigma_{zz} = \sigma_{rr} - (E^{-1} - E^2)\Sigma_1 + (E^{-2} + D^2E^{-1}r^2 - E)\Sigma_{-1} \\ \sigma_{\theta z} = DEr(\Sigma_1 - E^{-1}\Sigma_{-1})\end{gathered} \tag{3.1}$$

The Rivlin's solution contains the constant A, which is found from the boundary condition $\sigma_{rr} = 0$ at $r = E^{-1/2}R_0$, in the case when all the material of the rod deforms elastically, and from the condition that the stress tensor remains continuous during the passage across the elastoplastic boundary (whose equation can be written by virtue of the symmetry of the problem, in the form $r = c$) in the case when the region of plastic flow exists.

Let us analyse the elastic solution. The orthonormal basis $\mathbf{l}', \mathbf{m}', \mathbf{n}'$ consisiting of the eigenvectors of the stress tensor, can be obtained as a result of rotating, by an angle ψ', the orthonormed basis $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ relative to the vector $\mathbf{e}_r$. The angle ψ' is given by the equation

$$\operatorname{tg}\psi' = 2DEr\,\{[(D^2r^2 + E^{-1} + E^2)^2 - 4E]^{1/2} + D^2r^2 + E^{-1} - E^2\}^{-1} \tag{3.2}$$

The principal stresses $\sigma_1, \sigma_2, \sigma_3$ are given in terms of the squares of the principal strains as follows:

$$\sigma_i = -p + v_i^2\Sigma_1 + v_i^{-2}\Sigma_{-1} \tag{3.3}$$
$$v_1^2 = E^{-1}, 2v_{2,3}^2 = D^2r^2 + E^{-1} + E^2 \pm [(D^2r^2 + E^{-1} + E^2)^2 - 4E]^{1/2}$$

It can be shown that the inequalities $v_2^2 > v_1^2 \geqslant v_3^2$, hold for the squares of the principal strains, therefore the modulus of the greatest tangential stress, by virtue of (3.3), will be equal to $^1/_2(\sigma_2 - \sigma_3)$ or

$$|\tau_{\max}| = {}^1/_2\,(\Sigma_1 - E^{-1}\Sigma_{-1})\,[(D^2r^2 + E^{-1} + E^2)^2 - 4E]^{1/2} \tag{3.4}$$

The quantity $|\tau_{\max}|$ is an increasing function of r. The greatest tangential stress acts on an area with the normal $\mathbf{v}' = \cos(\psi' - \pi/4)\,\mathbf{e}_z - \sin(\psi' - \pi/4)\,\mathbf{e}_\theta$. The value of the elongation E^* and torsion D^*, for which the greatest tangential stress at the contour of the cross-section of the rod first reaches its yield point k, is found from the following system of equations:

$$(E^{*3} - 1)\,[(^1/_2 + \beta)\,E^* + {}^1/_2 - \beta] = {}^1/_4 D^{*2}R_0^2\,[(^1/_2 + \beta)\,E^* + 1 - 2\beta]$$
$$(2k)^2\,(\Sigma_1 - E^{*-1}\Sigma_{-1})^{-2} = [(D^{*2}R_0^2 + 1)\,E^{*-1} + E^{*2}]^2 - 4E^*$$

It can be shown that the above system has a solution and its root E^* satisfies the inequality $E^* > 1$. The corresponding torsional moment has the value

$$M^* = {}^1/_2\pi D^* E^{*-1} R_0^4\,(\Sigma_1 - E^{*-1}\Sigma_{-1})$$

When the load is increased further, a region of plastic flow appears within the cross-section of the rod. We choose, as the load parameter, $\lambda = E$, in which case the twist, torsional moment and the radius of the elastic kernel will all be functions of E. Within the elastic region we have the Rivlin solution (3.1). The Euler velocity field has the following physical components:

$$v_r = -r/(2E),\ v_\theta = D^\cdot rz/E,\ v_z = z/E \tag{3.5}$$

The actual position of the elastoplastic boundary $r = c$, of the twist and the elongation, must satisfy the equation

$$(2k)^2\,(\Sigma_1 - E^{-1}\Sigma_{-1})^{-2} = (D^2c^2 + E^{-1} + E^2)^2 - 4E,\ E > E^* \tag{3.6}$$

which follows from (3.4), and the condition of continuity of the stress tensor at $r = c$.

We will make a series of assumptions concerning the stress and velocity fields in the plastic region, whose validity will be confirmed by constructing a solution of the problem based on these assumptions.

1) the orthonormalized basis $\mathbf{l}'', \mathbf{m}'', \mathbf{n}''$ of the eigenvectors of the stress tensor is obtained by rotating the basis $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ by a certain angle ψ'' about the vector $\mathbf{e}_r$, and the relation $\psi'' = \psi'$ must hold when $r = c$;

2) the angle ψ'' and the principal stresses $\sigma_1, \sigma_2, \sigma_3$ (as well as the principal elastic strains v_1^e, v_2^e, v_3^e) are functions of r and E only;

3) the radial component of the velocity v_r depends only on r and E, $v_z = 2\alpha(E)\,z$, $v_\theta = \gamma(E)\,rz$.

By virtue of the above assumptions we can write the physical non-zero components of the stress tensor in the following form:

$$\sigma_{rr} = \sigma_1,\ \sigma_{\theta\theta} = \sigma_2 - 2k\sin^2\psi'' \tag{3.7}$$
$$\sigma_{zz} = \sigma_2 - 2k\cos^2\psi'',\ \sigma_{\theta z} = k\sin 2\psi''$$

The equations of equilibrium now reduce to a single equation (the remaining equations are satisfied by virtue of the choice of hydrostatic pressure p /4/):

$$\partial\,(r\sigma_1)/\partial r = \sigma_2 - 2k\sin^2\psi'' \tag{3.8}$$

Integrating (3.8) we obtain

$$\sigma_1 = \int (\sigma_2 - \sigma_1 - 2k \sin^2 \psi'') r^{-1} dr \tag{3.9}$$

Using the defining relations (2.3), we can express the difference between the principal stresses $\sigma_2 - \sigma_1$ in terms of the principal elastic strains

$$\sigma_2 - \sigma_1 = (v_2^{e2} - v_1^{e2}) \Sigma_1 + (1/v_2^{e2} - 1/v_1^{e2}) \Sigma_{-1}$$

Substituting the last expression into (3.9) we conclude that the stress tensor in the plastic zone is determined by the principal elastic strains v_1^e, v_2^e and the angle ψ''. In order to determine the eight unknown functions $v_1^e, v_2^e, v_3^e, \psi'', v_r, v_\theta, v_z, \Lambda$, we obtain a system of six first-order partial differential equations from the tensor Eq.(2.5), and we must supplement them by the equation of incompressibility and the Tresca yield criterion

$$v_1^e v_2^e v_3^e = 1, \quad (v_2^{e2} - v_3^{e2}) \Sigma_1 + (1/v_2^{e2} - 1/v_3^{e2}) \Sigma_{-1} = 2k \tag{3.10}$$

We shall write the tensor Eq.(2.5) in the form of a system of scalar equations. The components of the tensor $\mathbf{F}^e, \mathbf{F}^{e-1}, \mathbf{F}^{e\cdot}, \mathbf{L}, \partial f/\partial \sigma$ are taken relative to the basis $\mathbf{l}'', \mathbf{m}'', \mathbf{n}''$. The matrices of the tensors $\mathbf{F}^e$ and $\partial f/\partial\sigma$ in this basis are diagonal

$$[\mathbf{F}^e] = \mathrm{diag}\,(v_1^e, v_2^e, v_3^e), \quad [\partial f/\partial \sigma] = \mathrm{diag}\,(0, 1, -1) \tag{3.11}$$

where $[\mathbf{A}]$ is the matrix of the tensor $\mathbf{A}$ in the basis $\mathbf{l}'', \mathbf{m}'', \mathbf{n}''$. We denote the elements of the matrix L_{ik} $(i, k = 1, 2, 3)$: by

$$\begin{gathered} L_{11} = \partial v_r/\partial r, \ L_{12} = -v_\theta \cos\psi''/r, \quad L_{13} = v_\theta \sin \psi''/r \\ L_{21} = \cos\psi'' \partial v_\theta/\partial r, \ L_{22} = v_r \cos^2\psi''/r + 2\alpha (E) \sin^2\psi'' + {}^1/_2 r\gamma (E) \times \\ \sin 2\psi'', \ L_{23} = [\alpha (E) - v_r/(2r)] \sin 2\psi'' + r\gamma (E) \cos^2 \psi'' \\ L_{31} = -\sin\psi'' \partial v_\theta/\partial r, \ L_{32} = [\alpha (E) - v_r/(2r)] \sin 2\psi'' - r\gamma (E) \times \\ \sin^2\psi'', \ L_{33} = v_r \sin^2\psi''/r + 2\alpha (E) \cos^2\psi'' - {}^1/_2 r\gamma (E) \sin 2\psi'' \end{gathered} \tag{3.12}$$

The physical components of the material derivative of the tensor $\mathbf{F}^e$ with respect to the load parameter E form a symmetric matrix and have the following form:

$$\begin{gathered} F_{rr}^{e\cdot} = (\partial/\partial E + v_r \partial/\partial r)\, v_1^e \\ F_{r\theta}^{e\cdot} = -v_\theta (v_2^e \cos^2\psi'' + v_3^e \sin^2\psi'' - v_1^e)/r \\ F_{rz}^{e\cdot} = -v_\theta (v_2^e - v_3^e) \sin 2\psi''/(2r) \\ F_{\theta\theta}^{e\cdot} = (\partial/\partial E + v_r \partial/\partial r)(v_2^e \cos^2\psi'' + v_3^e \sin^2\psi'') \\ 2F_{\theta z}^{e\cdot} = (\partial/\partial E + v_r \partial/\partial r)[(v_2^e - v_3^e)\sin 2\psi''] \\ F_{zz}^{e\cdot} = (\partial/\partial E + v_r \partial/\partial r)(v_2^e \sin^2\psi'' + v_3^e \cos^2\psi'') \end{gathered} \tag{3.13}$$

We denote the elements of the symmetric matrix $[\mathbf{F}^{e\cdot}]$ by $F_{ik}^{e\cdot}$ $(i, k = 1, 2, 3)$:

$$\begin{gathered} F_{11}^{e\cdot} = F_{rr}^{e\cdot}, \quad F_{12}^{e\cdot} = F_{r\theta}^{e\cdot} \cos\psi'' + F_{rz}^{e\cdot} \sin\psi'' \\ F_{13}^{e\cdot} = F_{rz}^{e\cdot} \cos\psi'' - F_{r\theta}^{e\cdot} \sin\psi'' \\ F_{22}^{e\cdot} = F_{\theta z}^{e\cdot} \sin 2\psi'' + F_{\theta\theta}^{e\cdot} \cos^2\psi'' + F_{zz}^{e\cdot} \sin^2\psi'' \\ F_{23}^{e\cdot} = F_{\theta z}^{e\cdot} \cos 2\psi'' - {}^1/_2 (F_{\theta\theta}^{e\cdot} - F_{zz}^{e\cdot}) \sin 2\psi'' \\ F_{33}^{e\cdot} = -F_{\theta z}^{e\cdot} \sin 2\psi'' + F_{\theta\theta}^{e\cdot} \sin^2\psi'' + F_{zz}^{e\cdot} \cos^2\psi'' \end{gathered} \tag{3.14}$$

Considering (2.5) as a matrix equation on the basis of the eigenvectors of the stress tensor $\mathbf{l}'', \mathbf{m}'', \mathbf{n}''$, taking into account formulas (3.11)-(3.14) and eliminating the multiplier Λ, we obtain the following system of five scalar equations:

$$\partial v_r/\partial r - F_{rr}^{e\cdot}/v_1^e = 0 \tag{3.15}$$

$$\begin{gathered} v_2^e v_3^e (v_r/r + 2\alpha) + (v_2^e - v_3^e) \sin 2\psi'' F_{\theta z}^{e\cdot} - (v_3^e \cos^2\psi'' + v_2^e \sin^2\psi'') F_{\theta\theta}^{e\cdot} - \\ (v_3^e \sin^2\psi'' + v_2^e \cos^2\psi'') F_{zz}^{e\cdot} = 0 \\ \cos\psi'' (v_1^{e2} \partial v_\theta/\partial r - v_2^{e2} v_\theta/r) - (v_1^e + v_2^e)(\cos\psi'' F_{r\theta}^{e\cdot} + \sin\psi'' F_{rz}^{e\cdot}) = 0 \\ \sin\psi'' (v_3^{e2} v_\theta/r - v_1^{e2} \partial v_\theta/\partial r) + (v_1^e + v_3^e)(\sin\psi'' F_{r\theta}^{e\cdot} - \cos\psi'' F_{rz}^{e\cdot}) = 0 \\ 2r\gamma (E)(v_3^{e2} \cos^2\psi'' - v_2^{e2} \sin^2\psi'') + (v_3^e + v_2^e)(2\alpha - v_r/r) \sin 2\psi'' - \\ (v_2^e + v_3^e)[(F_{zz}^{e\cdot} - F_{\theta\theta}^{e\cdot}) \sin 2\psi'' + 2\cos 2\psi'' F_{\theta z}^{e\cdot}] = 0 \end{gathered}$$

Direct substitution of the expression for $F_{r\theta}^{e^*}$ and $F_{rz}^{e^*}$ from (3.13) into the third and fourth equation of system (3.15) shows, that they are equivalent and can therefore be reduced to the following equation:

$$\partial v_\theta/\partial r - v_\theta/r = 0$$

The above relation is satisfied identically by virtue of assumption 3). Since $\operatorname{tr}[\partial f/\partial \sigma] = 0$, it follows that the equation of incompressibility $v_1^e v_2^e v_3^e = 1$ is equivalent to the equation

$$\partial v_r/\partial r + v_r/r = -2\alpha(E) \tag{3.16}$$

The condition that the axial velocity v_z is continuous at the elastoplastic boundary, yields $\alpha = (2E)^{-1}$. From (3.16) it follows that $v_r = -\alpha r$. Thus the velocity components v_r, v_z have the same form (3.5) in the plastic zone as in the region of the elastic kernel. It can be shown that the second equation of system (3.15) is satisfied identically after substituting into it the quantities $F_{\theta\theta}^{e^*}$, $F_{zz}^{e^*}$, $F_{\theta z}^{e^*}$ from (3.13). The first equation of system (3.15) and the condition of continuity at $r = c$ will be satisfied provided that we assume that $v_1^{e2} = E^{-1}$ within the plastic zone. Equation (3.10) will be determined by the squares of the elastic strains

$$v_2^{e2} = k\Psi + \Xi, \quad v_3^{e2} = -k\Psi + \Xi, \quad v_1^{e2} = E^{-1} \tag{3.17}$$
$$\Psi \equiv (\Sigma_1 - E^{-1}\Sigma_{-1})^{-1}, \; \Xi \equiv [k^2(\Sigma_1 - E^{-1}\Sigma_{-1})^{-2} + E]^{1/2}$$
$$(c(E) < r < R_0 E^{-1/2}, \; E > E^*)$$

After some reduction, we can write the last equation of system (3.15) in the form

$$r\partial\psi''/\partial r - 2E\partial\psi''/\partial E + 2r\gamma(E)E\{[P(E) - 1/2]\cos^2\psi'' - [P(E) + 1/2]\sin^2\psi''\} + 3P(E)\sin 2\psi'' = 0 \tag{3.18}$$

$$P(E) \equiv [(2k)^{-2} E\Psi^{-2} + 1/4]^{1/2}$$
$$(c(E) < r < R_0 E^{-1/2}, \; E > E^*)$$

The condition of continuity $\psi'' = \psi'$ where the angle ψ' is given by formula (3.2), must hold when $r = c$. We therefore have

$$\operatorname{tg}\psi''|_{r=c} = E(2\Xi - E^2 - E^{-1})^{1/2}(\Xi + k\Psi - E^2)^{-1} \tag{3.19}$$

The angle ψ'' represents a solution of the non-linear Cauchy problem (3.18), (3.19) for the first-order partial differential equation with free initial curve, whose integral can be found using the method of characteristics /8/.

4. ***Qualitative analysis of the process of elastoplastic deformation.*** We can carry out a qualitative analysis of the stress-strain state of a rod with torsion beyond the yield point, without the need to integrate the Cauchy problem (3.18), (3.19). We note that the loading process is not simple, since the angle ψ'' depends on E, and the principal axes of the stress tensor change their orientation as E increases. The stress state corresponds to the edge $\sigma_2 - \sigma_3 = 2k$ of the Tresca prism. The slip areas in the actual configuration have the normals $\mathbf{v}'' = \cos(\psi'' - \pi/4)\,\mathbf{e}_z - \sin(\psi'' - \pi/4)\,\mathbf{e}_\theta$. The slip surfaces represent the cylindrical co-axial surfaces $r = \text{const}$. The tangential stress vector $\boldsymbol{\tau}''$ acting on the slip area has the following physical components: $\tau_r'' = 0$, $\tau_\theta'' = k\cos(\psi'' - \pi/4)$, $\tau_z'' = k\sin(\psi'' - \pi/4)$. For this reason the slip lines, i.e. the lines tangent at every point to $\boldsymbol{\tau}''$, represent spiral lines wound around the cylinders $r = \text{const}$ at a constant angle $\psi'' - \pi/4$ (for each cylinder $r = \text{const}$). The radial compression of the material represents a pure elastic deformation. This, however, does not mean that the rod, which has been twisted beyond its yield point with the torsional load then removed, will return to its reference (natural) state. Indeed, the elastic part of the defromation is fully reversible only in the case of conceptual processes terminating in complete unloading.

Thus we can regard the complete elastoplastic deformation of a rod under torsion as the superposition of a pure elastic deformation caused by radial compression, and shear deformation along the spiral lines whose geometry is wholly determined by the angle ψ''. The plane cross-sections of the rod remain plane during the elastoplastic deformation. They are displaced along the z axis with velocity $v_z = z/E$, with simultaneous rotation about the z axis with an angular velocity of $\gamma(E)z$.

The peripheral component of the velocity v_θ is determined with the accuracy up to the multiplier $\gamma(E)$. If we admit the solutions in which the tangential component of velocity has a discontinuity along the slip surfaces, then $\gamma(E)$ will, in principle, be an arbitrary function of the load parameter. Therefore, the possibility arises of constructing, for every, arbitrary chosen function $\gamma(E)$, a consistent stress field which is continuous everywhere.

Calculating the torsional moment and axial force, we obtain

$$M = {}^1/_2\pi c^3 E\Psi^{-1}(2\Xi - E^2 - E^{-1})^{1/2} + 2\pi k \int\limits_c^{R_0E^{-1/2}} r^2 \sin 2\psi'' \, dr \tag{4.1}$$

$$N = \pi c^2 (E^2 - E^{-1})\Psi^{-1} + {}^1/_2\pi c^2 (E^{-1}\Sigma_{-1} - {}^1/_2\Sigma_1)(2\Xi - E^2 - E^{-1}) +$$
$${}^1/_2\pi (R_0{}^2E^{-1} - c^2)\{\Sigma_1(k\Psi + \Xi - E^{-1}) + \Sigma_{-1}[(k\Psi + \Xi)^{-1} - E]\} -$$
$$\pi k (R_0{}^2E^{-1} - c^2) - 6\pi k \int\limits_c^{R_0E^{-1/2}} r\cos^2\psi'' \, dr$$

The radius of the elastic kernel $r = c(E)$ is found from the equation $N = 0$. Therefore, the position of elastoplastic boundary and the torsional moment corresponding to the elongation E will depend on the choice of angular velocity $\gamma(E)z$ of material flow within the plastic zone. The problem of torsion cannot be solved within the framework of the theory of flow under finite deformation, since the principal directions of the stress tensor depend on the choice of velocity field.

Let us consider the critical state, when the whole material of the rod passes into the state of plastic flow. The value of the elongation $E = E_*$ at which the critical state is reached can be found as follows. We have, by virtue of the symmetry $\operatorname{ctg}\psi''|_{r=0} = 0$, and therefore from the initial condition (3.19) it follows that the limit elongation E_* is the root of the equation

$$-\Sigma_1E^4 + \Sigma_{-1}E^3 + 2kE^2 + \Sigma_1E - \Sigma_{-1} = 0 \tag{4.2}$$

which should exceed the value of E^* satisfying the inequality $E^* > 1$. Thus the quantity E_* which depends on the coefficients of the reaction Σ_1, Σ_{-1} and yield point k may serve as the criterion of the passage of the rod to its critical state under torsion. Passing to the limit in (4.1) as $E \to E_*$, we obtain

$$M_* = \int\limits_0^{R_0E_*^{-1/2}} r^2 \sin 2\psi''|_{E=E_*} \, dr \tag{4.3}$$

where E_* is the root of (4.2).

According to the theory of small deformations, the magnitude of the limiting torsional moment is found as follows /9/:

$$M_*^{\text{inf}} = 2/3\pi k R_0{}^3 \tag{4.4}$$

Estimating the integral on the right-hand side of (4.3) and using relation (4.4), we obtain the estimate

$$M_*/M_*^{\text{inf}} \leqslant E_*^{-3/2} < 1 \tag{4.5}$$

which can be used as the basis of the following assertion: the limiting torsional moment of a cylindrical rod of radius R_0 calculated from the geometrically non-linear theory of flow, is at least $E_*^{3/2}$ times smaller than the limiting torsional moment determined using the theory of flow under small deformations, provided that one and the same yield condition (2.1) is used in both theories.

5. *The complete solution of the elastoplastic problem of torsion.* The complete solution of the problem is given by formulas (3.7), (3.8), (3.10), (3.17) and (4.1). The angle ψ'' is given by the integral of the Cauchy problem (3.18), (3.19) and must be determined within a curvilinear triangle bounded by the curves $r = c(E)$, $r = R_0/E^{1/2}$, $E = E_*$ in the plane r, E. The Cauchy problem is formulated correctly, since the curves $r = R_0E^{-1/2}$, $r = 0$ represent the characteristics of the differential Eq.(3.18). After changing the variables

$$\xi = \ln r, \quad \eta = \ln E^{1/2}, \quad w = \ln \operatorname{tg}\psi'' + 6\int [(2k)^{-2}(e^{\eta}\Sigma_1 - e^{-\eta}\Sigma_{-1})^2 + {}^1/_4]^{1/2} \, d\eta$$

we can write (3.18) in the form

$$\partial w/\partial\xi - \partial w/\partial\eta + 2\gamma(e^{2\eta})e^{2\eta+\xi} \times \tag{5.1}$$
$$\{e^{h(\eta)}[P(e^{2\eta}) - {}^1/_2]e^{-w} - e^{-h(\eta)}[P(e^{2\eta}) + {}^1/_2]e^{w}\} = 0$$
$$h(\eta) = 6\int [(2k)^{-2}(e^{\eta}\Sigma_1 - e^{-\eta}\Sigma_{-1})^2 + {}^1/_4]^{1/2} \, d\eta$$

The integral $h(\eta)$ can be expressed in terms of elliptic integrals of the first and second kind as follows:

$$
\begin{gathered}
{}^1/_6 h(\eta) = ({}^1/_4 + 4a^2b^2)^{1/2}\,[\mathrm{F}(\varepsilon, \delta) - \mathrm{E}(\varepsilon, \delta) + \\
[q^2(\eta) + {}^1/_4]^{1/2}\,[q^2(\eta) - 4a^2b^2]^{1/2}/q(\eta)) \\
\varepsilon = \arccos\,[2ab/q(\eta)],\ \ \delta = (1 + 16a^2b^2)^{-1/2} \\
a^2 \equiv \Sigma_1/(2k),\ \ b^2 \equiv -\Sigma_{-1}/(2k),\ \ q(\eta) \equiv a^2e^{\eta} + b^2e^{-\eta}
\end{gathered}
$$

After changing to new variables ω, η $(\omega = \xi + \eta)$ and making the substitution $\chi = e^{\omega}$, Eq.(5.1) is reduced to an ordinary Ricatti differential equation (ω occurs in it as a parameter)

$$
\partial\chi/\partial\eta = 2e^{\omega}\gamma(e^{2\eta})\,e^{\eta}\,\{e^{h(\eta)}\,[P(e^{2\eta}) - {}^1/_2] - e^{-h(\eta)}\,[P(e^{2\eta}) + {}^1/_2]\,\chi^2\} = 0 \tag{5.2}
$$

Let us substitute the variable η according to the formula

$$
\varkappa = 2\int \gamma(e^{2\eta})\,e^{\eta - h(\eta)}\,[P(e^{2\eta}) + {}^1/_2]\,d\eta \tag{5.3}
$$

and write $\Omega(\varkappa) \equiv e^{2h(\eta)}\,[P(e^{2\eta}) - {}^1/_2]/[P(e^{2\eta}) + {}^1/_2]$, where η should be replaced by the inversion of the integral (5.3). Then (5.2) becomes

$$
e^{-\omega}\partial\chi/\partial\varkappa = \Omega(\varkappa) - \chi^2 \tag{5.4}
$$

The function $\Omega(\varkappa)$ is found from the angular velocity of material flow in the plastic zone.

The general integral of the Ricatti Eq.(5.4), and hence the integral of the Cauchy problem (3.18), (3.19), can be obtained using the method of quadratures, if at least one particular solution is known. It is best to seek this particular solution in the form of a series in powers of $\varkappa$ (we can always shift the origin of the expansion if necessary)

$$
\chi = \sum_{n=0}^{\infty} \chi_n\varkappa^n, \quad \chi_0 = 0, \quad \chi_n = \chi_n(\omega); \quad \Omega = \sum_{n=0}^{\infty} \Omega_n\varkappa^n
$$

The coefficients of the expansion χ_n are given by the following recurrence formula:

$$
\begin{gathered}
(m+1)\,e^{-\omega}\chi_{m+1} = \Omega_m - \sum_{p=1}^{m-1} \chi_p\chi_{m-p}, \quad m = 2, 3, \ldots \\
\chi_1 = e^{\omega}\Omega_0, \quad \chi_2 = {}^1/_2 e^{\omega}\Omega_1
\end{gathered}
$$

The formal series in powers of $\varkappa$ representing the integral $\chi(\varkappa, \omega)$ of the Ricatti Eq. (5.4), will also be the actual solution of the Ricatti equation by virtue of the Cauchy theorem on the existence and uniqueness of an analytic solution of a differential equation with analytic right-hand side (see e.g. /10/), provided that $\Omega(\varkappa)$ is an analytic function of $\varkappa$. The last condition is satisfied if the angular velocity $\gamma(e^{2\eta})$ is a positive analytic function of η. Indeed, in this case there exists an analytic function (inversion of the integral (5.3)) $\eta = \eta(\varkappa)$, and since $h(\eta)$, $P(e^{2\eta})$ are analytic functions of η, it follows that the function $\Omega(\varkappa)$ will also be analytic. It also follows from the Cauchy theorem mentioned above, that the recurrence relations for χ_m converge.

The solution of the problem of the torsion of a cylindrical rod given here is complete, since the stress tensor as well as the velocity field in the elastic and plastic zones are all determined.

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ON THE FORMULATION OF THE CONTACT PROBLEM OF ELASTIC PLASTICITY*

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A differential and a variational formulation of the problem of contact interaction between an elastic-plastic body and a rigid support are examined. Equations of the theory of plastic flow with isotropic hardening, which is a particular modification of the Il'yushin theory of elastic-plastic processes /1, 2/, are taken as governing relationships. A proof is presented of the existence and uniqueness of the generalized solution. To simplify the description the problem is considered in a Cartesian rectangular system of coordinates.

Contact problems with governing relationships of the deformation theory of plasticity are presented in /3/. Variational formulations utilizing generalized governing relationships of plasticity are formulated in /4, 5/. However, the constraints mentioned there on the generalized governing relationships are obviously inadequate for the uniqueness of the soluton.

1. Differential formulation of the problem. A quasistatic strain process is considered for an elastic-plastic body occupying a domain Ω in R^3 with a smooth boundary S. It is assumed that the displacements and the gradients of the displacements are small, and consequently, the squares of the gradients as well as the rotations of body elements can be neglected, and the connection to the side of the rigid support is considered to be ideal and unilateral. The problem is formulated in a reference system fixed with respect to the rigid support. The Mises plasticity condition is taken as the loading surface.

It is assumed that the domain under investgation Ω can consist of two parts at each instant: $\Omega^e = \{\mathbf{x} \in \Omega \mid \sigma_i(\mathbf{x}) < \sigma_T\}$ and $\Omega^p = \{\mathbf{x} \in \Omega \mid \sigma_i(\mathbf{x}) = \sigma_T\}$. Here σ_i is the stress intensity. Material strain occurs elastically in the domain Ω^e; in the general case the domain Ω^p consists of an active loading zone Ω^{pa} and an unloading zone Ω^{pr}, not known in advance and to be determined.

The conditions governing the above-mentioned zones have the form ($f = \sigma_i - \sigma_T,\ g_{ij} = \partial f/\partial\sigma_{ij}$):

$$\text{if } \mathbf{x} \in \Omega^p \text{ and } g_{ij}dS_{ij} \leqslant 0, \text{ then } \mathbf{x} \in \Omega^{pr},$$
$$\text{if } \mathbf{x} \in \Omega^p \text{ and } g_{ij}dS_{ij} > 0, \text{ then } \mathbf{x} \in \Omega^{pa}.$$

We write the governing relationships in the domain Ω as

$$dS_{ij} = 2G(de_{ij} - d\lambda\, g_{ij}), \quad d\sigma = K\, d\varepsilon \tag{1.1}$$

where $d\sigma$, $d\varepsilon$ are increments of the mean pressure and the mean strain. The scalar factor $d\lambda$ equals zero in the domains Ω^e and Ω^{pr}. In the domain Ω^{pa}

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